

M(atrix) Theory on T_5/Z_2 Orbifold and Five-Branes ¹

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Abstract

We study M(atrix) theory description of M theory compactified on T_5/Z_2 orbifold. In the large volume limit we show that M theory dynamics is described by $\mathcal{N} = 8$ supersymmetric $USp(2N)$ M(atrix) quantum mechanics. Via zero-brane parton scattering, we show that each orbifold fixed point carries anomalous G-flux $\oint[G/2\pi] = -1/2$. To cancel the anomalous G-flux, we introduce twisted sector consisting of sixteen five-branes represented by fundamental representation hypermultiplets. In the small volume limit we show that M theory dynamics is effectively described by $(5+1)$ -dimensional $(8,0)$ supersymmetric $USp(2N)$ chiral gauge theory. We point out that both perturbative and global gauge anomalies are cancelled by the sixteen fundamental representation hypermultiplets in the twisted sector. We show that M(atrix) theory is capable of turning on spacetime background with the required sixteen five-branes out of zero-brane partons as bound-states. We determine six-dimensional spacetime spectrum from the M(atrix) theory for both untwisted and twisted sectors and find a complete agreement with the spectrum of $(2,0)$ supergravity. We discuss M(atrix) theory description of compactification moduli space, symmetry enhancement thereof as well as further toroidal compactifications.

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1 Introduction

Witten [1] has made an important observation that in the strong coupling limit all known superstring theories are most accurately described by M-theory. The M-theory is a theory with manifest eleven-dimensional Lorentz invariance and, when truncated to massless excitations, reduces to the eleven-dimensional supergravity with appropriate gravitational corrections to the Chern-Simons coupling. While enormous progress has been achieved based on analysis at the supergravity level, a satisfactory microscopic description of the M-theory was missing.

Recently, Banks *et.al.* [2] have put forward an interesting nonperturbative proposal of light-front M-theory, so-called M(atr)ix theory. By extrapolating Type IIA string into strong coupling regime and boosting infinitely along the ‘quantum’ M-theory direction, they have identified constituent partons with zero-branes and Chan-Paton gauge fields [3]. Light-front dynamics of these partons is governed by $\mathcal{N} = 16$ supersymmetric M(atr)ix quantum mechanics with $SU(N)$ gauge group and $SO(9)$ R-symmetry [4]. Implicit to their proposal are two crucial assumptions: non-anomalous $SO(10, 1)$ Lorentz invariance and existence of a threshold bound-state of the zero-brane partons. Despite these yet-unproven dynamical assumptions and intrinsically light-front description, the M(atr)ix theory has successfully passed numerous consistency checks so far. Furthermore, for finite N , the M(atr)ix theory is shown to correspond to discrete light-cone quantization (DLCQ) of M-theory [5].

Among the most important issues in M(atr)ix theory is a comprehensive understanding of nontrivial compactifications. So far, toroidal compactifications have been investigated mostly [6]. Being a perfectly smooth manifold, however, the toroidal compactifications may not reveal hidden intrinsically quantum and non-perturbative M(atr)ix theory aspects. In fact, it is not unlikely that intrinsically M(atr)ix theoretic aspects are tied with compactifications on nontrivial spaces. Within the approach based strictly on low-energy supergravity, the strongly coupled $E_8 \times E_8$ heterotic string has already provided such an example [7]: the strong coupling limit of heterotic string is described by the M-theory compactified on an orbifold $\mathbf{S}_1/\mathbf{Z}_2$. Located at the two orbifold fixed points are nine-branes or ‘end of the worlds’. The E_8 gauge multiplets arise as spectra of the twisted sector localized on the nine-brane world-volume and are the essential elements for satisfying the M-theory field equations. If it were to offer a non-perturbative definition of the M-theory, the M(atr)ix theory should provide an equally consistent quantum mechanical definition of $\mathbf{S}_1/\mathbf{Z}_2$ orbifold compactification and produce the same result as the supergravity analysis at the least. In the previous paper [8], we have investigated this issue and have found M(atr)ix theory definition of the orbifold compactification as well as the correct twisted sector spectra. Furthermore, based on the same M(atr)ix theory compactification, various dualities of the heterotic – Type I strings have been understood in detail [9].

In this paper, we extend our earlier work on M(atr)ix theory on orbifold and study $\mathbf{T}_5/\mathbf{Z}_2$ compactification in depth. Witten [10] and Dasgupta and Mukhi [11] have studied the same orbifold compactification of low-energy eleven-dimensional supergravity and have shown that the compactification vacua give rise to anomaly-free $(2, 0)$ supergravity spectra at low-energy. Despite the analysis was strictly based on the eleven-dimensional supergravity on a singular orbifold, their results have revealed a very interesting non-perturbative aspect of the M theory. Located at the 2^5 orbifold fixed points are anomalous half-units of G-flux for integral cohomology class $[G/2\pi] = -1/2$, where $G_{[MNPQ]} \equiv \nabla_{[M} C_{NPQ]}$ is the four-form tensor field strength of M theory. In order to maintain vanishing total G-flux on $\mathbf{T}_5/\mathbf{Z}_2$, sixteen five-branes has to be

put on $\mathbf{R}_{5,1}$. This gives rise to sixteen $(2,0)$ tensor multiplets as the twisted sector spectrum. The total spectrum turns out anomaly-free and in fact agrees with the spectrum of type IIB string compactified on $K3$.

Given that the result of Witten and of Dasgupta and Mukhi has already revealed non-perturbative aspects of the M theory, a complete account of $\mathbf{T}_5/\mathbf{Z}_2$ orbifold compactification should therefore serve as a highly nontrivial test to the M(atrrix) theory itself. Moreover, if the M(atrrix) theory is a complete description of M theory compactification as well as dynamics on it, fully consistent *spacetime background* including nontrivial five-branes on it as well as the complete *spacetime* spectrum should emerge dynamically out of the M(atrrix) theory. As we will show, this indeed turns out to be the case. In particular, we show that the needed sixteen five-branes are dynamically arranged as bound-states of zero-brane partons. This is precisely along the spirit of the M(atrrix) theory proposal: Spacetime background is a coherent configuration of M theory massless excitations. If the latter are to be described by zero-brane bound-states, so should be the case for the spacetime backgrounds.

In Section 2, we begin with M(atrrix) theory description of M theory compactified on a large $\mathbf{T}_5/\mathbf{Z}_2$ orbifold. By analyzing the action of orbifold projection to the Chan-Paton factors and R-symmetry, we show that the M(atrrix) theory relevant for the $\mathbf{T}_5/\mathbf{Z}_2$ compactification is given by $\mathcal{N} = 8$ supersymmetric M(atrrix) quantum mechanics with gauge group $\mathrm{USp}(2N)$ where N denotes the number of zero-brane partons. Via parton scattering off an orbifold fixed point, we probe presence of anomalous G-flux of $-1/2$ units emanating from each fixed points. Consistency of the orbifold compactification requires cancellation of total -16 units of anomalous G-flux. This determines uniquely the twisted sector spectrum to consist of thirty-two ‘half’ hypermultiplets in fundamental representation of $\mathrm{USp}(2N)$.

In Section 3, we study M(atrrix) theory description in the opposite limit, viz. M theory compactified on a small $\mathbf{T}_5/\mathbf{Z}_2$ orbifold. The M(atrrix) theory relevant in this limit may be derived in two different ways. One can start from the $\mathcal{N} = 8$ M(atrrix) quantum mechanics and T-dualize along all five orbifold directions. The method has been utilized previously for the case of a small $\mathbf{S}_1/\mathbf{Z}_2$ orbifold compactification, and exactly the same procedure applies in the present context. Alternatively, one may start with a covering space torus \mathbf{T}_5 in the small volume limit. The limit is described by $\mathcal{N} = 16$ supersymmetric Yang-Mills theory on a dual torus $\tilde{\mathbf{T}}_5$. One then apply the \mathbf{Z}_2 orbifold projection to obtain orbifold M(atrrix) theory relevant for small volume limit. Since the first approach is a straightforward exercise of the $\mathbf{S}_1/\mathbf{Z}_2$ orbifold compactification, we explain the second approach in more detail. We show that the prescription gives rise to $(0,8)$ supersymmetric $\mathrm{USp}(2N)$ chiral gauge theory on the dual torus $\tilde{\mathbf{T}}_5$ with one antisymmetric tensor representation hypermultiplet plus twisted sector consisting of thirty-two ‘half’ fundamental representation hypermultiplets. With these twisted sector spectrum, we show that the M(atrrix) gauge theory is free from perturbative gauge and nonperturbative global anomalies. Via the T-duality of the first approach, we then show that the the M(atrrix) gauge theory is nothing but the world-volume gauge theory of N overlapping small instantons of $\mathrm{SO}(32)$ Type I string theory. We also show that, degenerate $\mathbf{T}_5/\mathbf{Z}_2$ orbifold limit in which one of the five radii shrinks to zero, the M(atrrix) gauge theory is better interpreted as N overlapping small instantons of $E_8 \times E_8$ heterotic string theory.

One of the physically most appealing features of the M(atrrix) theory is that all space-time physical excitations are built out of bound-states of zero-brane partons. In particular, the sixteen longitudinal five-branes necessary for a consistent orbifold compactification can be

achieved via Landau-orbiting zero-brane parton configurations. In section 4, we show how the five-branes can be arranged and how the twisted sector spectrum is induced by the five-brane background. We then deduce $\mathbf{R}_{5,1}$ *spacetime* massless excitations as well as BPS spectra based entirely on the M(atrrix) theory construction. We find a complete agreement with the (2,0) chiral supergravity multiplet structure and the BPS spectra symmetry $\text{SO}(5,20; \mathbf{Z})$.

In section 5, we study further toroidal compactification, viz. M theory compactification on $\mathbf{T}_5/\mathbf{Z}_2 \otimes \mathbf{T}_m$. We show that the small volume limit is described by a M(atrrix) gauge theory on a dual orbifold $\tilde{\mathbf{T}}_5 \otimes \tilde{\mathbf{T}}_m/\mathbf{Z}_2$ in which bulk theory is coupled to ‘impurity’ boundary theory. In the bulk, the M(atrrix) gauge theory is given by $(5+m+1)$ -dimensional $\mathcal{N} = 16$ supersymmetric Yang-Mills theory of gauge group $\text{U}(N)$. Near the thirty-two orbifold fixed points, the M-theory dynamics is described by $(5+1)$ -dimensional $(0,8)$ supersymmetric $\text{USp}(2N)$ chiral gauge theories. The thirty-two twisted sector ‘half’ fundamental representation hypermultiplets are distributed among the thirty-two $(5+1)$ -dimensional chiral gauge theories. Different distribution corresponds to T-dual configuration of turning on Wilson lines either in heterotic $E_8 \times E_8$ or Type I $\text{SO}(32)$ string theories. Consistency of local description requires cancellation of $(5+1)$ -dimensional gauge anomalies at each orbifold boundaries. We show that this is achieved via the Wess-Zumino term and anomaly inflows from the bulk thereof.

2 M(atrrix) Theory on a Large $\mathbf{T}_5/\mathbf{Z}_2$ Orbifold

We begin with a large volume limit of the $\mathbf{T}_5/\mathbf{Z}_2$ orbifold. In this case, one should be able to describe the M theory parton dynamics accurate enough by a $(0+1)$ -dimensional M(atrrix) theory. However, the \mathbf{Z}_2 involution is not acting freely on \mathbf{T}_5 but generates $2^5 = 32$ fixed points. As such, as in string theories, nontrivial effects of the orbifold compactification are expected to be present as one approaches any of these orbifold fixed points.

In this section we show that, near the orbifold fixed point where the orbifold geometry is locally $\mathbf{R}_5/\mathbf{Z}_2$, M-theory parton dynamics is described by $\mathcal{N} = 8$ supersymmetric $\text{USp}(2N)$ M(atrrix) quantum mechanics. Moreover, via parton scattering off the orbifold fixed point, we probe the presence of anomalous magnetic G-flux $[G/2\pi]$ of $-1/2$ unit emanating from each orbifold fixed point. Since $\mathbf{T}_5/\mathbf{Z}_2$ is a compact space with $2^5 = 32$ fixed points, extra background carrying $+16$ units of magnetic G-flux has to be turned on in order for the orbifold compactification to be compatible with G-flux conservation. The needed compensating fluxes are provided precisely by turning on sixteen longitudinal five-branes on the noncompact spacetime $\mathbf{R}_{5,1}$. These sixteen longitudinal five-branes in turn affects the zero-brane parton dynamics in such a way manifestly consistent with M(atrrix) theory is defined. We show that the effects of sixteen longitudinal five-branes are represented by twisted sector consisting of thirty-two ‘half’ fundamental representation hypermultiplets of $\text{USp}(2N)$.

2.1 M(atrrix) Theory on $\mathbf{R}_5/\mathbf{Z}_2$ via Orbifold Projection

We begin by recalling general definition of M(atrrix) theory on a large orbifold [8]. In M(atrrix) theory, orbifold compactification is defined in terms of appropriate involution. From the covering space point of view, the orbifold is defined by starting with N original and N ‘image’ 0-brane partons on the covering space, then subsequently projecting out with involution. Therefore, in this description, coordinate and spinor matrices are $2N \times 2N$ in size. Consider the fixed point

of $\mathbf{R}_5/\mathbf{Z}_2$ located at $X_5 = X_6 = \dots = X_9 = 0$. The Chan-Paton condition acting on M(atr) theory coordinates is a discrete gauge symmetry transformation, which is a direct product of orientation reversal and parity transformation on \mathbf{R}_5 . The condition is given by [8]:

$$\begin{aligned} \mathbf{X}_{\parallel} &= + M \cdot \mathbf{X}_{\parallel}^T \cdot M^{-1}, \\ \mathbf{X}_{\perp} &= - M \cdot \mathbf{X}_{\perp}^T \cdot M^{-1}, \\ \Theta &= \Gamma_{\perp} M \cdot \Theta^T \cdot M^{-1}, \end{aligned} \quad \Gamma_{\perp} = \Gamma_5 \cdots \Gamma_9. \quad (1)$$

In M(atr) theory, the two-branes are Landau-orbiting collective excitations of the zero-brane partons. The two-brane charge is defined by the first Chern class $\mathcal{Z}_2 := [F/2\pi] = c_1(G)$ of gauge group G . As such, the above Chan-Paton condition inverts the two-brane orientation, hence, sign of the charge for every $x_{\perp} = (x_5, \dots, x_9)$ directions. In terms of eleven-dimensional supergravity, the flip of the first Chern class $[F/2\pi]$ corresponds to a simultaneous reversal $C_{\text{MNP}} \rightarrow -C_{\text{MNP}}$ for every $\text{MNP} = (5, \dots, 9)$.

The matrix M relates the original and the ‘image’ 0-brane partons. Hermiticity of the coordinate and the spinor matrices require $M^T \cdot M = \pm \mathbf{I}_{2N \times 2N}$. Possible choices of M compatible with this condition can be written as $M_{2N \times 2N} = \mathbf{I}_{N \times N} \otimes \sigma_{\mu}$. In the previous work [8], we have shown that there are two possible choices with orbifold conditions and embedding of the gauge symmetries. For $\mu = 0, 1, 3$, the compatible gauge group of the M(atr) theory turns out $\text{SO}(2N) \subset \text{SU}(2N)$, while for $\mu = 2$, it was $\text{USp}(2N) \subset \text{SU}(2N)$. In both choices of the M(atr) theory gauge group, the components \mathbf{X}_{\perp} ’s transform as adjoint representation, viz. two-index antisymmetric for $\text{SO}(2N)$ and symmetric for $\text{USp}(2N)$. We combine them with the M(atr) theory gauge field A_0 and define a gauge field:

$$A_M = (A_0, \dots, A_5) \equiv (A_0, \mathbf{X}_5, \mathbf{X}_6, \dots, \mathbf{X}_9). \quad (2)$$

The parallel components \mathbf{X}_{\parallel} transform as two-index symmetric representations for $\text{SO}(2N)$ and anti-symmetric representations for $\text{USp}(2N)$ respectively. We choose the M(atr) theory gauge group to be $\text{USp}(2N)$. In what follows, through various consistency checks, we will find that this is indeed the correct choice of the M(atr) gauge group.

The spinor field Θ decomposes into two eigenstates of Γ_{\perp} . Accordingly, the R-symmetry is broken to $\text{Spin}(4) \otimes \text{Spin}(5) \subset \text{Spin}(9)$. We adopt a representation of the sixteen-dimensional gamma matrices so that the chirality is chosen

$$\Gamma^{(11)} \equiv \Gamma^0 \Gamma^1 \Gamma^2 \cdots \Gamma^9 = 1. \quad (3)$$

In the decomposition of $\text{Spin}(5) \otimes \text{Spin}(4) \subset \text{Spin}(9)$ we represent the 16×16 gamma matrices as:

$$\begin{aligned} \Gamma^i &= \mathbf{I}_{4 \times 4} \otimes (\gamma_i)_{4 \times 4} & (i = 1, 2, 3, 4) \\ \Gamma^{i+4} &= (\gamma_i)_{4 \times 4} \otimes (\gamma_5)_{4 \times 4} & (i = 1, 2, 3, 4, 5) \end{aligned} \quad (4)$$

where

$$\begin{aligned} \gamma_i &= \begin{pmatrix} 0 & +\sigma_i \\ -\sigma_i & 0 \end{pmatrix}, & (i = 1, 2, 3) \\ \gamma_4 &= \begin{pmatrix} \mathbf{I} & 0 \\ 0 & \mathbf{I} \end{pmatrix}, & \gamma_5 = \begin{pmatrix} +i\mathbf{I} & 0 \\ 0 & -i\mathbf{I} \end{pmatrix}. \end{aligned} \quad (5)$$

These choices are made for consistency with the chirality convention so that

$$\gamma_1 \cdots \gamma_5 = 1, \quad \gamma_5^4 = 1. \quad (6)$$

Fermions are decomposed into two inequivalent multiplets. Decomposing the R-symmetry $\text{Spin}(4) = \text{SU}(2) \times \text{SU}(2)$, we denote \mathbf{S}_a and $\mathbf{S}_{\dot{a}}$ as the right-handed and the left-handed spinors and Dirac spinor of $\text{Spin}(5)$ at the same time. Therefore, the sixteen-component Majorana spinor can be decomposed into two inequivalent chiral spinors of the R-symmetry:

$$\Theta = (\mathbf{1}, \mathbf{2}, \mathbf{4}) \oplus (\mathbf{2}, \mathbf{1}, \mathbf{4}) \equiv \mathbf{S}_a \oplus \mathbf{S}_{\dot{a}}, \quad (7)$$

where

$$\begin{aligned} \Gamma_{\perp} \mathbf{S}_a &= -\mathbf{S}_a \\ \Gamma_{\perp} \mathbf{S}_{\dot{a}} &= +\mathbf{S}_{\dot{a}}. \end{aligned} \quad (8)$$

From the last condition in Eq. (1), we then find that \mathbf{S}_a transforms as an adjoint representation of $\text{USp}(2N)$ M(atrix) gauge group, while $\mathbf{S}_{\dot{a}}$ transforms as an antisymmetric representation.

The fact that the two eight-component spinors transform differently under the M(atrix) gauge group entails certain mismatch at one-loop. Indeed, as we will show in the next subsections, the orbifold M(atrix) quantum mechanics is consistent only after a twisted sector is included. The twisted sector consists of thirty-two hypermultiplets transforming as ‘half’ fundamental representations under the gauge group $\text{USp}(2N)$.

The field content of M(atrix) quantum mechanics is then given by:

Sector	Multiplet	Bosons	Fermions	$\text{Spin}(5) \otimes \text{Spin}(4)$	$\text{USp}(2N)$
untwisted	gauge	A_I	\mathbf{S}_a	$(\mathbf{5}, \mathbf{1}); (\mathbf{4}, \mathbf{2}_L)$	$2N(2N+1)/2$
	antisymmetric	\mathbf{X}^i	$\mathbf{S}_{\dot{a}}$	$(\mathbf{1}, \mathbf{4}); (\mathbf{4}, \mathbf{2}_R)$	$2N(2N-1)/2$
twisted	fundamental	ϕ_A^i	$\chi_{\dot{a}A}$	$(\mathbf{1}, \mathbf{4}); (\mathbf{4}, \mathbf{2}_R)$	$2N$

The untwisted sector of the orbifold M(atrix) theory Lagrangian is given by

$$\begin{aligned} L_{\text{untwisted}} = \text{Tr} \Big(& \frac{1}{2R} (D_t \mathbf{X}^i)^2 + \frac{1}{2R} (D_t A^I)^2 + \mathbf{S}_a D_t \mathbf{S}_a + \mathbf{S}_{\dot{a}} D_t \mathbf{S}_{\dot{a}} \\ & + \frac{R}{4} [\mathbf{X}^i, \mathbf{X}^j]^2 + \frac{R}{4} [A^I, A^J]^2 + \frac{R}{2} [A^I, \mathbf{X}^j]^2 \\ & + 2i R \mathbf{X}^i \sigma_{a\dot{a}}^i \{ \mathbf{S}_a, \mathbf{S}_{\dot{a}} \} - R \mathbf{S}_a \gamma_I [A^I, \mathbf{S}_a] + R \mathbf{S}_{\dot{a}} \gamma_I [A^I, \mathbf{S}_{\dot{a}}] \Big). \end{aligned} \quad (9)$$

The γ_I matrices act on $\text{Spin}(5)$ R-symmetry indices, which are suppressed in the expression.

Likewise, the twisted sector Lagrangian is given by

$$\begin{aligned} L_{\text{twisted}} = \Big[& \frac{1}{R} |(D_t + A_I) \phi_A^i|^2 + \bar{\chi}_A (D_t + R A_I \gamma^I) \chi_A \\ & + \frac{R}{2} \bar{\phi}_A^i [\mathbf{X}^i, \mathbf{X}^j] \phi_A^j - R |\bar{\phi}_A^i \phi_B^i|^2 \\ & - R \bar{\phi}_A^i \mathbf{S}_a \sigma_{a\dot{a}}^i \chi_{\dot{a}A} - R \bar{\chi}_{\dot{a}A} \sigma_{\dot{a}a}^i \mathbf{S}_a \phi_A^i \Big]. \end{aligned} \quad (10)$$

The $A, B = 1, \dots, 16$ indices denote the sixteen fundamental representation hypermultiplets obtained by adjoining pairs of ‘half’ fundamental representation hypermultiplets.

The corresponding Hamiltonian is straightforwardly derived:

$$\begin{aligned}
H_{\text{total}} = R \Big[& \text{Tr} \Big(+ \frac{1}{2} \mathbf{\Pi}_i^2 + \frac{1}{2} E_I^2 \\
& - \frac{1}{4} [\mathbf{X}^i, \mathbf{X}^j]^2 - \frac{1}{4} [A_I, A_J]^2 - \frac{1}{2} [A_I, \mathbf{X}^i]^2 \\
& - 2i \mathbf{X}^i \sigma_{a\dot{a}}^i \{ \mathbf{S}_a, \mathbf{S}_{\dot{a}} \} + \mathbf{S}_a \gamma_I [A^I, \mathbf{S}_a] - \mathbf{S}_{\dot{a}} \gamma_I [A^I, \mathbf{S}_{\dot{a}}] \\
& + |\Pi_{iA}|^2 - \bar{\chi}_A A_I \gamma^I \chi_A \\
& - \frac{1}{2} \bar{\phi}_A^i [\mathbf{X}^i, \mathbf{X}^j] \phi_A^j + |\bar{\phi}_A^i \phi_B^i|^2 \\
& + \bar{\phi}_A^i \mathbf{S}_a \sigma_{a\dot{a}}^i \chi_{\dot{a}A} + \bar{\chi}_{\dot{a}A} \sigma_{\dot{a}a}^i \mathbf{S}_a \phi_A^i \Big) \Big]. \tag{11}
\end{aligned}$$

The overall dependence on the 11-th directional radius R represent correctly that Eq.(11) should be interpreted as the light-cone Hamiltonian.

2.2 Probing Anomalous G-Flux via Parton Scattering

The $\mathbf{T}_5/\mathbf{Z}_2$ orbifold has $2^5 = 32$ fixed points. It is likely that these fixed points are more than mere geometrical singular loci. Indeed, for $\mathbf{S}_1/\mathbf{Z}_2$ orbifold [8], we have found that the each fixed point carries $2^3 = 8$ units of D8-brane charge and create nonzero vacuum energy tadpole. In order to cancel the RR charge and the vacuum energy, it was necessary to introduce eight complex fermions in the fundamental representation of the gauge group.

We now examine similar possibility for the $\mathbf{T}_5/\mathbf{Z}_2$ orbifold fixed points using the zero-brane partons as probes. Consider a neighborhood of one of the 32 fixed points. Locally the geometry is $\mathbf{R}_5/\mathbf{Z}_2$, an orbifold of noncompact five-dimensional space. Place a zero-brane parton as a local probe. A zero-brane parton scattering slowly off the fixed point will experience presence of a mirror zero-brane parton moving opposite to the probing zero-brane parton. As such we expect that the net effect of orbifold fixed point is equivalent to a potential generated via forward scattering between the two zero-branes. Dynamics of a zero-brane moving toward the fixed point is described by $\text{USp}(2)$ M(atrix) quantum mechanics. Consider the zero-brane parton moving in (8,9) plane with velocity v and impact parameter b relative to the orbifold fixed point. This is described by the background field configuration $X_8 = vt \sigma_3/2$ and $X_9 = b \sigma_3/2$. Expanding around the background, it is straightforward to see that bosonic and spinor fields in the rank-two antisymmetric representation are flat directions, reflecting translational invariance on $\mathbf{R}_{5,1}$. Bosonic and spinor fields in the adjoint (symmetric) representation give rise to massive fluctuations. In the covariant background field gauge, there are two bosonic modes with frequency $\gamma^2 \tau^2 + b^2$, two bosonic modes with frequency $\gamma^2 \tau^2 + b^2 \pm 2\gamma$. Similarly, there are eight massive fermionic modes. Thus, the one-loop amplitude is given by

$$\begin{aligned}
[\mathcal{A}_{\text{D0-O4}}]_{\text{USp}}(\gamma\tau, b) = & \det^{-2}(-\partial_\tau^2 + \gamma^2 \tau^2 + b^2) \cdot \det^{-1}(-\partial_\tau^2 + \gamma^2 \tau^2 + b^2 + 2\gamma) \\
& \cdot \det^{-1}(-\partial_\tau^2 + \gamma^2 \tau^2 + b^2 - 2\gamma) \cdot \det^{+4} \begin{pmatrix} \partial_\tau & \gamma\tau - ib \\ \gamma\tau + ib & \partial_\tau \end{pmatrix} \tag{12}
\end{aligned}$$

Again, straightforward calculation yields the result as

$$[\mathcal{A}_{\text{D0-O4}}]_{\text{USp}}(r, v) = v \int_0^\infty \frac{ds}{\sqrt{\pi s}} e^{-sr^2} \frac{1 + \cos 2vs - 2 \cos vs}{\sin vs}, \quad (13)$$

where $r^2 = b^2 + v^2 \tau^2$. Expanding the integrand in the small velocity limit, we get the potential energy for a slow zero-brane parton scattering off the orbifold fixed point (4-orientifold):

$$[\mathcal{V}_{\text{D0-O4}}]_{\text{USp}}(r, v) = -\frac{v^2}{2r^3} + \frac{25}{32} \frac{v^4}{r^7} + \dots \quad (14)$$

Suppose the M(atrix) gauge group were $\text{SO}(2N)$ instead of $\text{USp}(2N)$. In this case, the zero-brane parton scattering off the fixed point is described by untwisted sector one-loop amplitude of $\text{SO}(2)$ M(atrix) quantum mechanics. In this case, massive modes are provided by the symmetric representations. There are four bosonic massive modes with frequency $\gamma^2 \tau^2 + b^2$ and eight fermionic massive modes. The one-loop amplitude in this case is given by

$$[\mathcal{A}_{\text{D0-O4}}]_{\text{SO}}(\gamma\tau, b) = \det^{-4}(-\partial_\tau^2 + \gamma^2 \tau^2 + b^2) \cdot \det^{+4} \begin{pmatrix} \partial_\tau & \gamma\tau - ib \\ \gamma\tau + ib & \partial_\tau \end{pmatrix} \quad (15)$$

Straightforward regularized calculation yields

$$[\mathcal{A}_{\text{D0-O4}}]_{\text{SO}}(r, v) = v \int_0^{+\infty} \frac{ds}{\sqrt{\pi s}} e^{-sr^2} \frac{2 - 2 \cos vs}{\sin vs}. \quad (16)$$

We thus find the effective potential

$$[\mathcal{V}_{\text{D0-O4}}]_{\text{SO}}(r, v) = +\frac{v^2}{2r^3} + \frac{5}{32} \frac{v^4}{r^7} + \dots \quad (17)$$

Let us contrast the differences between SO and USp choices of the M(atrix) gauge group. For both cases, massive modes that contribute to the one-loop amplitude, hence, to the effective potential arise from the two-index symmetric representation. For $G=\text{USp}(2N)$ this is the adjoint representation supermultiplet and is interpreted as the zero-brane parton coordinates on $\mathbf{T}_5/\mathbf{Z}_2$. For $G=\text{SO}(2N)$ this is the symmetric representation hypermultiplet and is interpreted as the transverse coordinates of zero-brane partons on noncompact spacetime $\mathbf{R}_{5,1}$. Since we have arranged the scattering to take place entirely on $\mathbf{T}_5/\mathbf{Z}_2$ but at a fixed position on $\mathbf{R}_{5,1}$, by translational invariance, the nontrivial one-loop effect should arise entirely from the fluctuations of $\mathbf{T}_5/\mathbf{Z}_2$ part coordinates of the zero-brane. This is precisely the case if we have chosen the M(atrix) gauge group as $\text{USp}(2N)$ but not for $\text{SO}(2N)$. This argument provides one of the physical basis for our choice of $\text{USp}(2N)$ as the correct M(atrix) theory gauge group.

We now consider parton scattering off a longitudinal five-brane background. The minimally charged five-brane is represented by a hypermultiplet transforming as a fundamental representation under the M(atrix) theory gauge group². For both $\text{SO}(2N)$ and $\text{USp}(2N)$ gauge group choices, a simple counting of the massive bosonic and fermionic modes in the background of

²Likewise, multiply charged longitudinal five-branes are represented by hypermultiplets in higher-dimensional representations of the M(atrix) gauge group. In M(atrix) theory, these hypermultiplets in arbitrary representation can be constructed by turning on appropriate backgrounds of multiple longitudinal five-branes on $\mathbf{R}_{5,1}$ spacetime. We will show the construction explicitly in section 4.2.

$X_8 = v\sigma_3/2$ and $X_9 = b\sigma_3/2$ shows that the one-loop scattering amplitude is given by the same expression. For a unit-charged longitudinal five-brane background, it is

$$\mathcal{A}_{D0-L5}(\gamma\tau, b) = \det^{-4}(-\partial_\tau^2 + \frac{\gamma^2\tau^2}{4} + \frac{b^2}{4}) \cdot \det^{+4} \begin{pmatrix} \partial_\tau & \frac{\gamma\tau}{2} - i\frac{b}{2} \\ \frac{\gamma\tau}{2} + i\frac{b}{2} & \partial_\tau \end{pmatrix} \quad (18)$$

This yields the potential energy

$$\mathcal{V}_{D0-L5}(r, v) = +\frac{v^2}{r^3} + \frac{5}{4}\frac{v^4}{r^7} + \dots \quad (19)$$

Comparing the potential energy exerted by an orbifold fixed point (four-orientifold) Eq. (14) to that exerted by a unit-charged longitudinal five-brane Eq. (19), we find that the orbifold fixed point carries positive one-half or negative one-half units of the five-brane tension depending on whether M(atrix) gauge group is $SO(2N)$ or $USp(2N)$. Since the brane charges are measured in units of tension, we interpret this result as implying that the orbifold fixed point of $SO(2N)$ respectively $USp(2N)$ M(atrix) theory carries $\pm 1/2$ units of five-brane charge. The $-1/2$ unit of anomalous G-flux emanating from the orbifold fixed point was first pointed out by Witten [10] and, independently, by Dasgupta and Mukhi [11] based on spacetime gravitational anomaly cancellation consideration. It is gratifying that the M(atrix) quantum mechanics can probe the anomalous G-flux correctly and consistently with their results. In fact, the M(atrix) theory provides another means of probing the anomalous G-flux via gauge anomaly cancellation, hence, confirming results obtained by Witten [10] and Dasgupta and Mukhi [11]. In the next section, we will also find that the $USp(2N)$ is the correct choice of the M(atrix) gauge group from perturbative and global gauge anomaly cancellation considerations in the shrinking $\mathbf{T}_5/\mathbf{Z}_2$ limit.

Recalling that there are $2^5 = 32$ orbifold fixed points on $\mathbf{T}_5/\mathbf{Z}_2$, we find that, only for the $USp(2N)$ M(atrix) gauge group choice but not for $SO(2N)$, the total anomalous G-flux of $32 \times (-1/2) = -16$ units can be cancelled by turning on sixteen longitudinal five-branes. Locally the cancellation is incomplete unless the $\mathbf{T}_5/\mathbf{Z}_2$ orbifold is taken to degenerate limit in the compactification moduli space. For now, we will tacitly assume such a limit should do consistency of the compactification arise. We will elaborate more on this issue in section 5.

Note that, in deducing the anomalous G-flux of $-1/2$ unit from each orbifold fixed point, hence, total thirty-two multiplicity of twisted sector hypermultiplets, seemingly trivial $1/2$ factor difference of $U(1)$ charge between the fundamental and the two-index representations has played a crucial role. See Eqs.(12, 18). While this is a direct consequence of a simple group theoretic fact, it is instructive to understand the $1/2$ factor difference from geometrical point of view as well. Let us recall the procedure for calculating zero-brane parton scattering off the orbifold fixed point and off a static object represented by the hypermultiplets. In the first case, using the method of image, we have arranged a mirror zero-brane parton at the \mathbf{Z}_2 transformed position. The net potential is then calculated in terms of relative distance and relative velocity between the original zero-brane parton and the mirror zero-brane. Note that this procedure yields *twice* the distance between the original zero-brane parton and the orbifold fixed position. At the same time the relevant mass that enters the relative dynamics is the reduced mass, viz. *half* of the original zero-brane parton mass. In the case of zero-brane parton scattering off the fundamental representation hypermultiplet, we have arranged the hypermultiplet as a fixed background. Thus, the distance as well as the reduced mass that enters the relative dynamics are geometrically measured ones.

Therefore, if we denote the relative distance, velocity and potential as R, V, U between the probe zero-brane parton and the image parton and as r, v, \mathcal{V} between the probe zero-brane parton and the orbifold fixed point, then the equation of motion for the probing zero brane reads

$$\frac{1}{2}\ddot{R} = -\frac{dU(R, V)}{dR} \quad (20)$$

in the first method of description and

$$\ddot{r} = -\frac{d\mathcal{V}(r, v)}{dr} \quad (21)$$

in the second case. Since the two descriptions should be the same, using the fact that $R = 2r, v = \dot{x} = V/2$, we find that

$$\mathcal{V}(r, v) = \frac{1}{2}U(R = 2r, V = 2v). \quad (22)$$

This explains the group theory factor $1/2$ difference in Eqs.(12, 18) more intuitively via elementary geometric consideration. In the above results, we have taken the normalization that r, v denote the relative distance and velocity for the fundamental representation, viz. a single longitudinal five-brane, and $2r, 2v$ for the two-index adjoint or antisymmetric representations.

3 M(atrix) Gauge Theory on Small $\mathbf{T}_5/\mathbf{Z}_2$

In this section, we study M theory compactification for small $\mathbf{T}_5/\mathbf{Z}_2$ orbifold. The first method is to make Fourier transform the M(atrix) quantum mechanics we have identified above. This is essentially to take into account of all possible open string configurations connecting among the zero-brane partons and their images around the compactified directions. Since this is a straightforward extension of the method utilized already for $\mathbf{S}_1/\mathbf{Z}_2$ compacification [8], in this section, we explain alternative prescription for deriving the same result.

3.1 M(atrix) Gauge Theory via Orbifold Projection

We begin with M theory compactification on a small \mathbf{T}_5 . In the M(atrix) theory approach to M theory, this limit is most appropriately described by $\mathcal{N} = 16$ supersymmetric $U(N)$ Yang-Mills theory on a dual torus $\tilde{\mathbf{T}}_5$. M theory dynamics is then extracted from scattering of excitations in the moduli space of the gauge theory. Next consider M theory compactification on a small $\mathbf{T}_5/\mathbf{Z}_2$. The \mathbf{Z}_2 involution is a combined operation of parity transformation to the \mathbf{T}_5 coordinates and orientation reversal $C_{\text{MNP}} \rightarrow -C_{\text{MNP}}$. Accordingly, the M(atrix) gauge theory is obtained via \mathbf{Z}_2 involution of the $\mathcal{N} = 16$ supersymmetric $U(N)$ Yang-Mills theory.

We begin with arranging mirror-image partons on \mathbf{T}_5 . The total number of zero-brane partons is equal to $2N$. Their dynamics is described by $\mathcal{N} = 16$ $U(2N)$ Yang-Mills theory on a dual torus $\tilde{\mathbf{T}}_5$. In the vanishing volume limit, the field content consists of a gauge multiplet $(A_\mu, \mathbf{S}_{\dot{a}})$ and a matter multiplet $(\mathbf{X}^i, \mathbf{S}_a)$. Acting on these fields, the \mathbf{Z}_2 involution is a combination of ‘parity’ transformation P and orientation reversal Ω . The ‘parity’ transformation acts as:

$$\begin{aligned} P : \quad A_\mu(x) &\rightarrow P \cdot A_\mu(x) \cdot P^{-1} \equiv +A_\mu(P \cdot x) \\ \mathbf{X}^i(x) &\rightarrow P \cdot \mathbf{X}^i(x) \cdot P^{-1} \equiv -\mathbf{X}^i(P \cdot x) \\ \Theta_\alpha(x) &\rightarrow P \cdot \Theta_\alpha(x) \cdot P^{-1} \equiv \Gamma_\perp \Theta(P \cdot x) . \end{aligned} \quad (23)$$

The orientation reversal transformation Ω acts as:

$$\begin{aligned}\Omega &: A_\mu(x) \rightarrow \Omega \cdot A_\mu(x) \cdot \Omega^{-1} \equiv \pm A_\mu^T(x) \\ \mathbf{X}^i(x) &\rightarrow \Omega \cdot \mathbf{X}^i(x) \cdot \Omega^{-1} \equiv \pm \mathbf{X}^{iT}(x) \\ \Theta_\alpha(x) &\rightarrow \Omega \cdot \Theta_\alpha^T(x) \cdot \Omega^{-1} \equiv \pm \epsilon \Theta^T(x),\end{aligned}\tag{24}$$

in which we have explicitly shown two-fold overall sign ambiguity and two-fold $\epsilon = \pm$ relative sign ambiguity in the definition of the orientation reversal transformation. In the present case, the latter ambiguity is resolved by an appropriate choice of the six-dimensional spinor chirality convention. The combined operations $\Pi \equiv \Omega \cdot P$ is a symmetry of the $U(2N)$ M(atrix) gauge theory, where the fields transform as:

$$\begin{aligned}A_\mu(x) &\rightarrow \pm A_\mu^T(P \cdot x) \\ \mathbf{X}^i(x) &\rightarrow \mp \mathbf{X}^{iT}(P \cdot x) \\ \Theta(x) &\rightarrow \pm \Gamma_\perp \Theta^T(P \cdot x).\end{aligned}\tag{25}$$

The combined operation yields the $\mathbf{T}_5/\mathbf{Z}_2$ orbifold M(atrix) gauge theory so long as $P \cdot x = x$ itself. As in the M(atrix) quantum mechanics, the involution acts both on the dual parameter space $\tilde{\mathbf{T}}_5$ and the gauge group $U(2N)$.

The two-fold overall sign ambiguity in Eq. (25) corresponds to two distinct choices of gauge group projected out of $SU(2N)$. For the upper sign choice, Eq. (25) projects out the gauge fields into two-index symmetric representation, the pseudoscalar fields into two-index antisymmetric representation and the sixteen component spinors into a sum of symmetric representation, eight component spinor \mathbf{S}_a and antisymmetric representation, eight component spinor $\mathbf{S}_{\bar{a}}$. Note that two spinors have opposite six-dimensional chirality. This is precisely the field content of a (5+1)-dimensional (0,8) supersymmetric $USp(2N)$ chiral gauge theory consisting of a gauge supermultiplet (A_I, \mathbf{S}_a) and one hypermultiplet $(\mathbf{X}^i, \mathbf{S}_{\bar{a}})$ in antisymmetric representation. For the lower sign choice, Eq.(25) yields $SO(2N)$ gauge group instead of $USp(2N)$. As we will see momentarily, this choice leads to an inconsistent theory and we only concentrate on the $USp(2N)$ gauge group choice from now on.

The above field content as it stands is anomalous. This is because the involution projects out (5+1)-dimensional fermion content in a chirally asymmetric manner: one adjoint fermion of positive chirality and one antisymmetric fermion of negative chirality. The (5+1)-dimensional perturbative gauge anomaly is given by quartic Casimir operator $C_4(\mathbf{R}) \equiv \text{Tr}_{\mathbf{R}} F^4$ of $USp(2N)$ gauge group. With the above field content we find that the anomaly measured in unit of left-handed fundamental representation fermion is given by

$$\begin{aligned}X_{\text{untwisted}}^8 &= \text{Tr}_{\text{adjoint}} F^4 - \text{Tr}_{\text{antisymm}} F^4 \\ &= \left[\{(2N+8)\text{tr} F^4 + 3(\text{tr} F^2)^2\} - \{(2N-8)\text{tr} F^4 + 3(\text{tr} F^2)^2\} \right] \\ &= 16 \text{tr} F^4.\end{aligned}\tag{26}$$

The gauge anomaly is purely quartic. As such, unless the anomaly is cancelled by adding appropriate chiral matter, the M(atrix) gauge theory is inconsistent. By inspection, it is clear that by adding sixteen fundamental or, equivalently, thirty-two ‘half’ fundamental representation hypermultiplets the nonfactorizable anomaly can be cancelled. Had the M(atrix) theory

gauge group been $SO(2N)$ instead of $USp(2N)$, the symmetric and the antisymmetric representations switch their role as gauge and adjoint representations. In turn, the resulting anomaly is minus to that of the $USp(2N)$ case. This anomaly cannot be cured neither by adding chiral matters nor by Green-Schwarz mechanism. The theory makes sense only if it is coupled to (0,1) supergravity in six dimension [18]³.

In fact, the $USp(2N)$ gauge group with one antisymmetric representation and thirty-two ‘half’ fundamental representation hypermultiplets is very special. This is the only (0,8) gauge theory content in six dimensions which is free from perturbative gauge anomalies for *all* N . Since the rank of gauge group N is interpreted as the total number of zero-brane partons, hence, the size of longitudinal momentum in units of $1/R$, we find that the anomaly consideration selects uniquely the $USp(2N)$ M(atrix) gauge group as a consistent choice. We also note that, when adjoined with $SO(32)$ gauge group of the twisted sector, the M(atrix) gauge theory remains anomaly free even after it is coupled to (0,1) supergravity in six dimensions [18].

One needs to check further consistency condition. For a single parton, $N=1$, the M(atrix) gauge theory group is $USp(2) = SU(2)$, and there arises potential inconsistency due to nonperturbative global gauge anomaly. This is because $\Pi_6(SU(2)) = \mathbf{Z}_{12}$. The consistency condition for absence of global gauge anomaly has been performed recently. For $USp(2) = SU(2)$, the number of fundamental representation hypermultiplets has to be 4 modulo 6 [19]. The sixteen fundamental, hence, thirty-two ‘half’ fundamental representations fits perfectly to the consistency condition. Putting together both the perturbative and global gauge anomaly consistency conditions, we thus conclude that the M(atrix) gauge theory with gauge group $USp(2N)$ describes a consistent M-theory compactification on $\mathbf{T}_5/\mathbf{Z}_2$ only when a twisted sector spectrum of thirty-two ‘half’ fundamental representation hypermultiplets are introduced. These hypermultiplets also exhibit $SO(32)$ global symmetry. Later they will be identified with the gauge group of sixteen longitudinal five-branes located at the orbifold fixed points.

Summarizing what we have found so far, field content and quantum numbers of the M(atrix) gauge theory are given by :

Sector	Multiplet	(Components)	$Spin(5) \otimes Spin(4)$	$USp(2N)$	$SO(32)$
untwisted	gauge	$(A_I; \mathbf{S}_a)$	$(\mathbf{5}, \mathbf{1}; \mathbf{4}, \mathbf{2}_L)$	$2N(2N + 1)/2$	1
	symmetric	$(\mathbf{X}^i; \mathbf{S}_{\dot{a}})$	$(\mathbf{1}, \mathbf{4}; \mathbf{4}, \mathbf{2}_R)$	$2N(2N - 1)/2$	1
twisted	fundamental	$(\phi_A^i; \chi_{\dot{a}A})$	$(\mathbf{1}, \mathbf{4}; \mathbf{4}, \mathbf{2}_R)$	$2N$	32

The untwisted sector Lagrangian on $\tilde{\mathbf{T}}_5$ is given by

$$\begin{aligned}
L_{\text{untwisted}} = \frac{1}{g_{\text{YM}}^2} \text{Tr} \int d^5\sigma \Big[& - \frac{1}{4} F_{\alpha\beta}^2 + (D_\alpha \mathbf{X}^i)^2 + \frac{1}{4} [\mathbf{X}^i, \mathbf{X}^j]^2 \\
& + \mathbf{S}_a \mathcal{D}_R \mathbf{S}_a + \mathbf{S}_{\dot{a}} \mathcal{D}_L \mathbf{S}_{\dot{a}} + 2i \mathbf{X}^i \sigma_{\dot{a}a}^i \{ \mathbf{S}_a, \mathbf{S}_{\dot{a}} \} \Big], \quad (27)
\end{aligned}$$

where the chiral covariant derivatives are $\mathcal{D}_L = D_t - \gamma_I D_I$ and $\mathcal{D}_R \equiv D_t + \gamma_I D_I$ respectively.

Similarly, the twisted sector Lagrangian is given by

$$L_{\text{twisted}} = \int d^5\sigma \Big[+ |D_\alpha \phi_A^i|^2 + \bar{\chi}_A \mathcal{D}_R \chi_A$$

³Closely related analysis but without consideration of potential global gauge anomalies has been made independently by [16, 17].

$$\begin{aligned}
& - \frac{1}{g_{\text{YM}}^2} |\bar{\phi}_A^i \phi_B^i|^2 - \bar{\phi}_A^i [\mathbf{X}^i, \mathbf{X}^j] \chi_A^j \\
& - \left[\bar{\phi}_A^i \sigma_{a\dot{a}}^i \mathbf{S}_a \gamma^i \chi_{\dot{a}A} - \phi_A^i \sigma_{\dot{a}a}^i \bar{\chi}_{\dot{a}A} \gamma^i \mathbf{S}_a \right].
\end{aligned} \tag{28}$$

3.2 Probing Anomalous G-Flux via T-Duality

It is possible to understand the M(atrrix) gauge theory constructed above in an intuitive way. In this sub-section, we show that the M(atrrix) gauge theory is in fact the world-volume gauge theory of N overlapping $\text{SO}(32)$ small instantons [23] of Type I string theory. This implies that dynamics of M-theory compactified on $\mathbf{T}_5/\mathbf{Z}_2$ orbifold can be understood via scattering experiments of gauge theory excitations in the moduli space of N overlapping $\text{SO}(32)$ small instanton world-volume gauge theory in the large N limit. Moreover, this also enables to identify the origin of the anomalous G-flux as T-dual of the nine-brane charge carried by orientifold in Type I string theory.

To proceed further, we take the 11-th direction as the ‘quantum’ direction. The ‘quantum’ direction is compactified on a circle of radius R_{11} along which the M-theory is boosted. The radius R_{11} will be taken to infinity in the end. Viewed this way, M(atrrix) theory compactified on transverse $\mathbf{T}_5/\mathbf{Z}_2$ is identified with Type IIA string theory compactified on \mathbf{T}_5 with 4-orientifolds (O4), which we will refer as Type \tilde{I}' theory.

On the compactified $\mathbf{T}_5/\mathbf{Z}_2$ are placed N D0-brane partons. The sixteen longitudinal five-branes become thirty-two D4-branes with \mathbf{Z}_2 symmetric identification. Label the compactified directions as X_5, \dots, X_9 and their radii as R_5, \dots, R_9 . Then the 4-orientifolds are located at $X_5 = 0, \pi R_5, \dots, X_9 = 0, \pi R_9$, viz. at the thirty-two fixed points of $\mathbf{T}_5/\mathbf{Z}_2$. Let us T-dualize all the compact directions. This maps the Type \tilde{I}' theory into Type I theory on a dual torus $\tilde{\mathbf{T}}_5$. The thirty-two D4-branes in the twisted sector turns into thirty-two D9-branes producing $\text{SO}(32)$ Type I string gauge group. Likewise, the N D0-brane partons turns into N parallel D5-branes wrapped around the dual torus.

As is well-known, due to the absence of Coulomb branch, these D5-branes are $\text{SO}(32)$ small instantons. Each individual small instantons carry $\text{USp}(2) = \text{SU}(2)$ world-volume gauge theory. When N small instantons overlap, the gauge symmetry is enhanced to $\text{USp}(2N)$. This is precisely the M(atrrix) gauge group we have identified earlier. Furthermore, the collective coordinates of small instantons along X_1, \dots, X_4 directions are identified with the non-compact transverse directions of infinitely boosted M-theory, viz, M(atrrix) theory. Due to Chan-Paton condition, it is known that the collective coordinates in a two-index, antisymmetric representation under $\text{USp}(2N)$ [23]. They are precisely the matter multiplet $(\mathbf{X}^i, \mathbf{S}_a)$ of the M(atrrix) gauge theory in the untwisted sector.

Finally, at the overlap of N D5-branes and thirty-two D9-branes, there appear thirty-two, ‘half’ fundamental representation matter multiplets of $\text{USp}(2N)$. Again they are precisely the twisted sector multiplets of the M(atrrix) gauge theory introduced in the previous section to ensure $\text{USp}(2N)$ gauge anomaly cancellation.

4 Massless Spacetime Spectrum

In this section, equipped with information of G-flux on the $\mathbf{T}_5/\mathbf{Z}_2$ orbifold fixed points, we determine the complete low-energy spectra that propagate on $R_{5,1}$. Recall that the M(atr)ix theory describes M theory infinitely boosted along $\mathbf{R}_{1,1} \in \mathbf{R}_{5,1}$. We follow the same strategy adopted in our previous work [8] for the heterotic M(atr)ix theory: we determine the low-energy spacetime spectrum by identifying all possible configurations of probing branes that are consistent with orbifold projection conditions.

On $\mathbf{R}_{5,1}$, massless fields are labelled by representations of the little group $\text{Spin}(4) := SU_L(2) \otimes SU_R(2)$. States of spin ≤ 2 are:

$$\begin{aligned}
\text{graviton} : & \quad (\mathbf{3}, \mathbf{3}) \\
\text{gravitino} : & \quad (\mathbf{3}, \mathbf{2}) \text{ or } (\mathbf{2}, \mathbf{3}) \\
\text{tensor} : & \quad (\mathbf{3}, \mathbf{1})_{\text{ASD}} \text{ or } (\mathbf{1}, \mathbf{3})_{\text{SD}} \\
\text{vector} : & \quad (\mathbf{2}, \mathbf{2}) \\
\text{spinor} : & \quad (\mathbf{2}, \mathbf{1}) \text{ or } (\mathbf{1}, \mathbf{2}) \\
\text{scalar} : & \quad (\mathbf{1}, \mathbf{1}).
\end{aligned} \tag{29}$$

The chiral $(2,0)$ supergravity in six dimensions, to which we will identify the M(atr)ix theory compactified on $\mathbf{T}_5/\mathbf{Z}_2$ below, admits two massless super-multiplets. The supergravity multiplet consists of

$$(\mathbf{3}, \mathbf{3}) \oplus 4(\mathbf{2}, \mathbf{3}) \oplus 5(\mathbf{1}, \mathbf{3}), \tag{30}$$

while tensor multiplet consists of

$$(\mathbf{3}, \mathbf{1}) \oplus 4(\mathbf{2}, \mathbf{1}) \oplus 5(\mathbf{1}, \mathbf{1}). \tag{31}$$

Below, we will identify all the bosonic field contents using the M(atr)ix theory branes as probes. From the $\mathcal{N} = 8$ supersymmetry, the fermionic fields content follows automatically and fits into the above multiplet structure.

We will also explore the moduli space

$$\mathcal{M}(\mathbf{T}_5/\mathbf{Z}_2) = SO(5, 21; \mathbf{Z}) \backslash SO(5, 21) / SO(5) \otimes SO(21). \tag{32}$$

One of the most interesting feature of M(atr)ix theory on $\mathbf{T}_5/\mathbf{Z}_2$ orbifold is that the required sixteen longitudinal five-branes can be constructed out of zero-brane partons. These longitudinal five-branes are BPS states of M(atr)ix theory [20] and reflect fully dynamical entities of the theory. The fact that we can arrange nontrivial *spacetime background* such as orbifold as well as excitations on it via composite bound-states of zero-brane partons should be viewed as the most interesting and important feature of the M(atr)ix theory.

4.1 Untwisted Sector of (2,0) Supergravity

Untwisted sector can be probed by studying the M(atr)ix theory far away from the fixed points. As in the heterotic M(atr)ix theory case, excitations of the off-diagonal parts of Eq.(1) are suppressed. Thus M(atr)ix theory in the untwisted sector is given by gauge group $G_{\text{untwisted}} = U(N) \otimes U(N)/\mathbf{Z}_2$, where the \mathbf{Z}_2 denotes orbifold action exchanging the two gauge groups.

Consider forward scattering in $\mathbf{R}_{5,1}$ between two zero-brane partons at impact parameter $|\mathbf{r}|$ and relative velocity \mathbf{v} . The scattering amplitude is easily calculated from the M(atrrix) Hamiltonian with gauge group $G_{\text{untwisted}}$. For toroidal compactification, the $2 \rightarrow 2$ scattering amplitude at M(atrrix) one-loop has been calculated by Berenstein and Corrado [21], and we gratefully make use of their results. With appropriate change of the normalization and using Euler-MacLaurin formula

$$\begin{aligned}\mathcal{A}_{2D0 \rightarrow 2D0} &= -6 \sum_{n_i} \int \frac{d\omega}{2\pi} \frac{\mathbf{v}^4}{(\omega^2/T_A + T_A|\mathbf{r}|^2 + T_A(n_i R_i)^2)} \\ &\approx -\frac{6}{T_A} \frac{d^5 y}{\text{Vol}(\mathbf{T}_5/\mathbf{Z}_2)} \int \frac{d\omega}{2\pi} \frac{\mathbf{v}^4}{(\omega^2 + T_A^2|\mathbf{r}|^2 + y^2)} \\ &= -\frac{\kappa_{11}^2}{2\pi^3 R_{11}^3} \frac{1}{R_5 R_6 R_7 R_8 R_9} \frac{\mathbf{v}^4}{|\mathbf{r}|^2}.\end{aligned}\tag{33}$$

Here, κ_{11} is the eleven-dimensional gravitational coupling constant with normalization $L_{11} = R^{(11)}/2\kappa_{11}^2 + \dots$ and $T_A^3 \equiv (2\pi)^4 R_{11}^3/\kappa_{11}^2$. The result is precisely the same as the $d = 6$ graviton-gravition elastic scattering amplitude at tree-level and at light-front frame. The scattering amplitude is \mathbf{Z}_2 -even, hence, survives the orbifold projection. This establishes the existence of $(\mathbf{3}, \mathbf{3})$ graviton propagating in $\mathbf{R}_{5,1}$. It is also possible that both of the zero-branes scatter in forward direction inside $\mathbf{T}_5/\mathbf{Z}_2$. The scattering amplitude, hence, the corresponding exchanged graviton state, is again \mathbf{Z}_2 -even. The $5 \cdot 6/2 = 15$ different graviton polarizations then give rise to $15(\mathbf{1}, \mathbf{1})$ real scalar fields on $\mathbf{R}_{5,1}$. Forward scattering between a zero-brane propagating on $\mathbf{R}_{5,1}$ and another propagating inside $\mathbf{T}_5/\mathbf{Z}_2$ is \mathbf{Z}_2 -odd. This implies that there is no Kaluza-Klein gravi-photon in the spectra.

As an another probe, we use the M(atrrix) two-brane made out of Landau-orbiting zero-brane partons. The two-brane can orient and propagate into various directions in $\mathbf{T}_5/\mathbf{Z}_2 \otimes \mathbf{R}_{5,1}$. Only \mathbf{Z}_2 invariant motions will be left after the orbifold projection. Consider a two-brane oriented entirely within $\mathbf{T}_5/\mathbf{Z}_2$ orbifold. Configuration of this so-called internal two-brane is given by [8]

$$A^I = \begin{pmatrix} Q & 0 \\ 0 & -Q^T \end{pmatrix}, \quad A^J = \begin{pmatrix} P & 0 \\ 0 & -P^T \end{pmatrix}, \quad (I, J = 5, \dots, 9) \tag{34}$$

where P, Q are $(N \times N)$ ($N \rightarrow \infty$) Hermitian matrices satisfying commutation relation $[Q, P] = +1$. As explained in Section 2, the \mathbf{Z}_2 parity is encoded into the BPS central charge density:

$$\mathcal{Z}^{IJ} \equiv [A^I, A^J] = [Q, P] \oplus [-Q^T, -P^T] = \sigma_3 \otimes \mathbf{I}_{N \times N}. \tag{35}$$

The configuration is \mathbf{Z}_2 -odd. To survive the orbifold projection, the internal two-brane has to propagate only inside $\mathbf{T}_5/\mathbf{Z}_2$. Geometrically there are $5 \cdot 6 \cdot 3/3 \cdot 2 = 10$ such distinct propagations. Coupled to each of them are 10 $(\mathbf{1}, \mathbf{1})$ real-valued scalar fields in the spacetime $\mathbf{R}_{5,1}$.

Next consider a two-brane extended across both $\mathbf{T}_5/\mathbf{Z}_2$ and $\mathbf{R}_{5,1}$. This is a direct counterpart of the twisted membranes in the heterotic M(atrrix) theory. The configuration is given by [8]:

$$A^I = \frac{1}{\sqrt{2}} \begin{pmatrix} Q & 0 \\ 0 & -Q^T \end{pmatrix}, \quad \mathbf{X}^j = \frac{1}{\sqrt{2}} \begin{pmatrix} P & 0 \\ 0 & P^T \end{pmatrix}, \quad (I = 5, \dots, 9; \quad j = 1, \dots, 4). \tag{36}$$

The BPS central charge density

$$\mathcal{Z}^{IJ} \equiv [A^I, \mathbf{X}^J] = \frac{1}{2}[Q, P] \oplus [-Q^T, P^T] = \frac{1}{2}\mathbf{I}_{2N \times 2N}, \quad (37)$$

shows that the configuration is \mathbf{Z}_2 -even. Therefore, orbifold projection surviving propagation is within $\mathbf{R}_{1,1}$, viz., BPS soliton strings in six dimensions. There are five distinct orientations of the two-brane inside $\mathbf{T}_5/\mathbf{Z}_2$. Since the strings couple to antisymmetric tensor fields, we find that the spacetime spectra include $5(\mathbf{5}, \mathbf{1}) \oplus 5(\mathbf{1}, \mathbf{3})$ (anti)-self-dual tensor fields.

Yet another possible probe is the longitudinal five-brane⁴ in the M(atrrix) theory. The configuration can be constructed again out of zero-brane partons in the M(atrrix) theory.

$$\begin{aligned} A^I &= \begin{pmatrix} Q_1 & 0 \\ 0 & -Q_1^T \end{pmatrix}, & A^J &= \begin{pmatrix} P_1 & 0 \\ 0 & -P_1^T \end{pmatrix}, & [Q_1, P_1] &= +1 \\ A^K &= \begin{pmatrix} Q_2 & 0 \\ 0 & -Q_2^T \end{pmatrix}, & A^L &= \begin{pmatrix} P_2 & 0 \\ 0 & -P_2^T \end{pmatrix}, & [Q_2, P_2] &= +1. \end{aligned} \quad (38)$$

The configuration carries the following BPS central charge densities

$$\begin{aligned} \mathcal{Z}^{IJ} &\equiv [A^I, A^J] = \frac{1}{2}\sigma_3 \otimes \mathbf{I}_{N \times N} & \mathcal{Z}^{KL} &\equiv [A^K, A^L] = \frac{1}{2}\sigma_3 \otimes \mathbf{I}_{N \times N} \\ \mathcal{Z}^{IJKL} &\equiv [A^I, A^J][A^K, A^L] = \mathcal{Z}^{IJ}\mathcal{Z}^{KL} = \frac{1}{4}\mathbf{I}_{2N \times 2N}. \end{aligned} \quad (39)$$

They show that all five distinct wrapping modes on $\mathbf{T}_5/\mathbf{Z}_2$ of the longitudinal five-brane are \mathbf{Z}_2 -parity even. Propagation only along $\mathbf{R}_{5,1}$ survives the orbifold projection and gives rise to five BPS soliton string in spacetime. Again coupled to them are 5 $(\mathbf{3}, \mathbf{1})$ and $(\mathbf{1}, \mathbf{3})$ (anti)-self-dual tensor fields⁵. However, these tensor fields are not independent ones. The longitudinal five-brane is electric-magnetic dual to the transverse two-brane in the M(atrrix) theory as is evidenced from Berry's phase analysis. The BPS strings on $\mathbf{R}_{5,1}$ reduced from these branes are electric-magnetic dual. In fact, (anti)-self-dual combinations of these BPS strings are the ones minimally coupled to the 5 $(\mathbf{3}, \mathbf{1})$ and 5 $(\mathbf{1}, \mathbf{3})$ tensor fields.

The above identification of 'electric' and 'magnetic' BPS strings on $\mathbf{R}_{5,1}$ point to an enhanced $SU(2)$ symmetry among tensor fields. A natural pairing of the 'electric' and the 'magnetic' BPS strings is that they are dual on \mathbf{T}_5 lattice. For a fixed volume of the $\mathbf{T}_5/\mathbf{Z}_2$, it is clear that the BPS tension of the (anti)-self-dual BPS string is given by:

$$T_{\mathbf{e}, \mathbf{m}} = \sum_{i=1}^5 \kappa |e^i R_i \pm \frac{\ell_{11}^2}{R_i} m^i|. \quad (40)$$

Thus, when the radius of any of the five toroidal direction is at the self-dual point $R_i = \ell_{11}$, the anti-self-dual strings become tensionless. Clearly, this is the M-theory counterpart of the momentum-winding duality. At the self-dual point, it then follows that there arises an enhanced $SU(2)$ symmetry.

⁴ It is the infinitely boosted D4-brane in strongly coupled type IIA string.

⁵ It is interesting to note that, while the longitudinal five-brane constructed as above carries extra two-brane charges $\mathcal{Z}^{IJ}, \mathcal{Z}^{KL}$, each of them are odd under \mathbf{Z}_2 -parity for the above propagation, hence, are completely projected out.

It is also possible that the longitudinal five-brane wraps less than four directions around $\mathbf{T}_5/\mathbf{Z}_2$. Inspection of the central charge density shows that propagation of these configurations are either \mathbf{Z}_2 -parity odd, hence, projected out or reduces to the above already analyzed cases.

Combining the $(\mathbf{3}, \mathbf{3})$ graviton and 5 $(\mathbf{3}, \mathbf{1})$ self-dual tensor fields, we obtain the $(0, 2)$ supergravity multiplet. Similarly, combining the remaining 5 $(\mathbf{1}, \mathbf{3})$ anti-self-dual tensor fields, 15 $(\mathbf{1}, \mathbf{1})$ scalars and 10 $(\mathbf{1}, \mathbf{1})$ scalars that couple to $\mathbf{T}_5/\mathbf{Z}_2$ propagation of the zero- and two-branes respectively, that couples to zero-brane propagation inside, we obtain five $(0, 2)$ tensor multiplets.

4.2 Turning on Longitudinal Five-Branes

In Section 2, we have shown that each orbifold fixed point carry $[G/2\pi] = -1/2$ G-flux. On $\mathbf{T}_5/\mathbf{Z}_2$, the total G-flux then adds up to -16 units. The anomalous flux has to be cancelled in order to maintain the $(0, 2)$ supersymmetry on $\mathbf{R}_{5,1}$. As we have discussed in Section 2, the cancellation can be achieved by putting sixteen longitudinal five-branes on $\mathbf{R}_{5,1}$. This is precisely M(atrix) theory counterpart to the earlier observation of Witten [10] and Dasgupta and Mukhi [11] within the context of low-energy eleven-dimensional supergravity approximation.

In M(atrix) theory, ground-state configuration of the zero-brane partons that corresponds to sixteen longitudinal five-branes is straightforwardly constructed. The configuration is essentially the same as Eq.(37) except that they are now oriented along $\mathbf{R}_{5,1}$. Labelling the sixteen five-branes as paired indices (a, b) where $a, b = 1, \dots, 16$, the zero-brane parton configuration on $\mathbf{R}_4 \in \mathbf{R}_{5,1}$ is given by $(32n + N) \times (32n + N)$ matrices:

$$\mathbf{X}^i = \begin{pmatrix} \mathbf{X}_{a=1}^i & & & & \\ & \mathbf{X}_{a=2}^i & & & \\ & & \ddots & & \\ & & & \mathbf{X}_{a=16}^i & \\ & & & & \mathbf{Y}^i \end{pmatrix}. \quad (41)$$

Here, \mathbf{X}_a denotes coordinate matrices for a -th longitudinal five-brane and its orientifold mirror five-brane in terms of $2n \times 2n$ matrices:

$$\begin{aligned} \mathbf{X}_a^1 &= \begin{pmatrix} Q_{1a} & 0 \\ 0 & Q_{1a}^T \end{pmatrix}, & \mathbf{X}_b^2 &= \begin{pmatrix} P_{1b} & 0 \\ 0 & P_{1b}^T \end{pmatrix}, & [Q_{1a}, P_{1b}] &= \delta_{ab} \frac{1}{\sqrt{2n}} \mathbf{I}_{2n \times 2n}; \\ \mathbf{X}_a^3 &= \begin{pmatrix} Q_{2a} & 0 \\ 0 & Q_{2a}^T \end{pmatrix}, & \mathbf{X}_b^4 &= \begin{pmatrix} P_{2b} & 0 \\ 0 & P_{2b}^T \end{pmatrix}, & [Q_{2a}, P_{2b}] &= \delta_{ab} \frac{1}{\sqrt{2n}} \mathbf{I}_{2n \times 2n}. \end{aligned} \quad (42)$$

The coordinates of N unclustered zero-brane partons and their images are separately denoted as \mathbf{Y}^i . In the infinite momentum limit of the M(atrix) theory, both n and N goes to infinity in proportion.

The configuration carries the correct BPS five-brane central charge of $+16$ units :

$$\begin{aligned} \mathcal{Z}^{[12][34]} &\equiv \sum_{a=1}^{16} \mathcal{Z}_a^{[12][34]} [\mathbf{X}^1, \mathbf{X}^2] [\mathbf{X}^3, \mathbf{X}^4] \\ &= 16 \frac{1}{2n} \mathbf{I}_{2n \times 2n}. \end{aligned} \quad (43)$$

Each of the sixteen configurations labelled $a = 1, \dots, 16$ give rise precisely to a longitudinal five-brane with a unit charge.

In addition, the configuration carries BPS two-brane central charges:

$$\begin{aligned}\mathcal{Z}_a^{[12]} &= [\mathbf{X}^1, \mathbf{X}^2]_a = \frac{1}{\sqrt{2n}} \mathbf{I}_{2n \times 2n}, & (a = 1, 2, 3, 4) \\ \mathcal{Z}_b^{[34]} &= [\mathbf{X}^3, \mathbf{X}^4]_b = \frac{1}{\sqrt{2n}} \mathbf{I}_{2n \times 2n}, & (b = 1, 2, 3, 4).\end{aligned}\quad (44)$$

As such, the two-brane BPS central charges diverge as $\sqrt{n} \rightarrow \infty$.

We now show that turning on longitudinal five-branes as above still preserves the $(2, 0)$ chiral supersymmetry on $\mathbf{R}_{5,1}$. To do so, consider a general longitudinal five-brane configuration Eqs.(41,42) in which the commutation relations of P_a, Q_b 's are now replaced by:

$$[Q_{1a}, P_{1b}] = \delta_{ab} A \mathbf{I}_{2n \times 2n}, \quad [Q_{2a}, P_{2b}] = \delta_{ab} B \mathbf{I}_{2n \times 2n}, \quad (45)$$

where $AB = \frac{1}{2n}$. Whether the configuration preserves the $(2, 0)$ chiral supersymmetry is most conveniently probed by scattering off a zero-brane. Consider the zero-brane probe moving slowly on $\mathbf{T}_5/\mathbf{Z}_2$:

$$A^1 = \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 0 & \\ & & & \ddots \\ & & & & r\sigma_3 \end{pmatrix}, \quad A^2 = \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 0 & \\ & & & \ddots \\ & & & & vt\sigma_3 \end{pmatrix}, \quad A^3 = A^4 = A^5 = 0. \quad (46)$$

For a single longitudinal five-brane (type IIA four-brane), the scattering has been studied by Lifschytz [27]. We generalize his calculation adopted to the present context. In the background Eqs.(41, 42, 46), massive states come from the last two rows and columns. They are four different stretched open strings connecting longitudinal five-branes on one hand and their images and the probe zero-brane and its image on another end. The one-loop integral gives rise to the phase shift for a zero-brane probe scattering off sixteen longitudinal five-branes is given by [27]:

$$\begin{aligned}\delta_{16 \text{ L5}}(r, v) &= - \int_{-\infty}^{+\infty} dt \mathcal{V}_{16 \text{ L5}}(r, vt) \\ &= 16 \int \frac{ds}{s} e^{-r^2 s} (8 \sinh sA \sinh sB \sin sv)^{-1} \\ &\times [2 + 2 \cos 2sv + 2 \cosh 2sA + 2 \cosh 2sB - 8 \cos sv \cosh sA \cosh sB].\end{aligned}\quad (47)$$

For $A \neq B$, this gives rise to *static* potential:

$$\mathcal{V}_{16 \text{ L5}}(r, v) \approx -16 \cdot \frac{1}{16} \frac{1}{r^3} \left[\frac{(A^2 - B^2)^2}{AB} + 2 \frac{(A^2 + B^2)}{AB} v^2 + \dots \right], \quad (48)$$

hence, incompatible with the $\mathcal{N} = 8$ supersymmetry of the orbifold M(atrrix) theory. Consequently, the $(2, 0)$ chiral supersymmetry on $\mathbf{R}_{5,1}$ is explicitly broken. On the other hand,

if $A = B = 1/\sqrt{2n}$ as in Eq.(42), the static potential vanishes identically and the potential between zero-brane and sixteen longitudinal five-branes is obtained. In fact, in this case $A = B = 1/\sqrt{2n}$, the scattering phase shift can be written as:

$$\delta_{16 \text{ L5}}(r, v) = +16 \int \frac{ds}{s} e^{-r^2 s} \left[\frac{1 - \cos sv}{\sin sv} + \frac{1}{4 \sinh^2 s / \sqrt{2n}} \frac{3 - 4 \cos sv + \cos 2sv}{\sin sv} \right]. \quad (49)$$

The first term is precisely the contribution of longitudinal five-brane charge $\mathcal{Z}^{[12][34]}$, while the second term is the contribution of $\mathcal{O}(N)$ zero-branes. Note that the effect of embedded two-brane charges $\mathcal{Z}^{[12]}, \mathcal{Z}^{[34]}$ to the phase shift vanishes miraculously in the limit $A = B$, corresponding to isotropic longitudinal five-branes.

The fact that the symmetric configuration $A = B$ is compatible with the $(2, 0)$ chiral supersymmetry on $\mathbf{R}_{5,1}$ can be also seen from the supersymmetry transformation rules of the M(atr)ix theory. In the background of the sixteen longitudinal five-branes,

$$\delta \Theta = 16 \cdot \frac{i}{4} (\Gamma_{12} A + \Gamma_{34} B) \epsilon = 0. \quad (50)$$

Therefore, only for the symmetric configuration $A = B$, we find a half-supersymmetry-preserving BPS condition

$$\Gamma_{[12]} = -\Gamma_{[34]} \quad \leftrightarrow \quad \Gamma_1 \Gamma_2 \Gamma_3 \Gamma_4 = +1 \quad \leftrightarrow \quad \Gamma_5 \Gamma_6 \cdots \Gamma_9 = +1. \quad (51)$$

The condition is precisely the same spinor projection condition for the $\mathbf{T}_5/\mathbf{Z}_2$ orbifold and for the $(2, 0)$ chiral supersymmetry on $\mathbf{R}_{5,1}$. We conclude that turning on sixteen longitudinal five-branes symmetrically as in Eqs.(41,42) is compatible with the supersymmetry.

We again emphasize the importance of above microscopic construction of longitudinal five-brane background. This is in complete agreement with the spirit of the M(atr)ix theory: nontrivial spacetime backgrounds as well as localized excitations propagating on it are to be built entirely out of zero-brane partons. That this is a consistent picture should be hardly surprising. The M-theory graviton is a composite bound-state of zero-brane partons. Nontrivial spacetime background is nothing but a coherent state configuration of the gravitons, hence, in turn, composite bound-state of zero-brane partons at the microscopic level. Our construction of the consistent $\mathbf{T}_5/\mathbf{Z}_2$ background then should be viewed as a realization of nontrivial spacetime out of matrices.

4.3 Twisted Sector of (2,0) Supergravity

In Section 3, we have already identified the twisted sector degrees of freedom in the M(atr)ix theory from the consideration of the G-flux conservation and gauge anomaly cancellation of $\mathcal{N} = 1$ supersymmetric gauge theories on a dual torus $\tilde{\mathbf{T}}_5$. Having turned on sixteen longitudinal five-branes and cancelled gauge and supersymmetry anomalies, we now determine the twisted sector spacetime spectrum.

So far we have not specified positions of parallel longitudinal five-branes on $\mathbf{T}_5/\mathbf{Z}_2$. Associated with each longitudinal five-brane are five coordinates. They transform as $(\mathbf{5}, \mathbf{1})$ under $SO(5) \otimes SO(4)$. Therefore, there are 80 real scalar fields on $\mathbf{R}_{5,1}$. Associated also with each longitudinal five-brane is an anti-self-dual tensor field. Combining them together, we find sixteen tensor multiplets. Excitations of the sixteen longitudinal five-branes are described by open

two-branes connecting them. Consider the five-branes clustered near one of the 32 fixed points. Positions of the sixteen five-branes on $\mathbf{T}_5/\mathbf{Z}_2$ are given by:

$$A^I = \begin{pmatrix} r_1\sigma_3 \otimes \mathbf{I} & & & \\ & r_2\sigma_3 \otimes \mathbf{I} & & \\ & & \ddots & \\ & & & r_{16}\sigma_3 \otimes \mathbf{I} \\ & & & & 0 \end{pmatrix} \quad (52)$$

Here, σ_3 denotes pairing of five-branes with their images across the fixed point. In this background, fluctuations of the longitudinal five-branes are described by the off-diagonal parts of the \mathbf{X}^i ($i = 1, 2, 3, 4$) matrices.

What are the possible singularities and enhanced symmetries thereof? Recall that we have chosen $\mathrm{USp}(2N)$ as the gauge group of the M(atr)ix theory on $\mathbf{T}_5/\mathbf{Z}_2$.⁶ Inferring from the canonical matrix structure, we find that there arises two possible enhanced symmetries in the twisted sector, which turns out to be the direct counterpart of A_{m-1} and D_m singularities $m \leq 16$ of type IIB string compactified on K3.

The A_{m-1} singularities arise when the five-branes coalesce on a point on $\mathbf{T}_5/\mathbf{Z}_2$ away from the fixed point. In this case, the off-diagonal elements in the diagonal sub-matrices become massless. Since the diagonal sub-matrices are unconstrained Hermitian matrices, we find that the enhanced symmetry is $(U(m) \times U(m))/\mathbf{Z}_2$, where each of the two $U(m)$'s are associated with overlapping five-branes and their orientifold images, and the \mathbf{Z}_2 projects out into \mathbf{Z}_2 -parity even configurations of the two clusters. The overlap configuration is achieved by tuning $5m - 5$ parameters. Hence, the loci of A_{m-1} type singularity and associated $U(m)$ enhanced symmetry is a hypersurface of co-dimension $5(17 - m)$.

Similarly the D_m singularities arise when the five-branes coalesce on one of the fixed points. In this case, in addition to the off-diagonal elements in the block-diagonal Hermitian matrices, the off-diagonal Hermitian, antisymmetric matrices become massless and excited. Since all of the off-diagonal entries of \mathbf{X}^i are excited, the resulting enhanced symmetry is given by the group in which \mathbf{X}^i are in the adjoint representations. Recall that we have chosen the M(atr)ix theory gauge group to be $\mathrm{USp}(2N)$. In this case, the \mathbf{X}^i 's are in antisymmetric representations of $\mathrm{USp}(2N)$. Hence, we identify that the symmetry group of which the \mathbf{X}^i 's are in adjoint representations is $\mathrm{SO}(2m)$. We find that the enhanced symmetry when m ($m \leq 16$) longitudinal five-branes approach one of the fixed points there arises an enhanced global symmetry group of D_m type, $\mathrm{SO}(2m)$. This is precisely what is known to take place for type IIB string compactified on K3 with D_m type singularities.

5 Further Toroidal Compactifications

It is of interest also to study higher-dimensional compactification in which $\mathbf{T}_5/\mathbf{Z}_2$ forms a subspace. The simplest example is a direct product compactification $(\mathbf{T}_5/\mathbf{Z}_2) \otimes \mathbf{T}_n$ for $n = 1, 2, 3, 4$. They all possess $\mathcal{N} = 8$ supersymmetries. What would be corresponding M(atr)ix theory description to these compactifications?

⁶ Note that background configurations of transverse two-branes and longitudinal five-branes are independent of the choice of the gauge groups. It is only when we examine enhanced symmetries that distinguishes the $\mathrm{SO}(2N)$ and $\mathrm{USp}(2N)$.

5.1 M(atr)ix Gauge Theory on Dual Orbifold $\tilde{\mathbf{T}}_5 \otimes (\tilde{\mathbf{T}}_n/\mathbf{Z}_2)$

The starting point is the M(atr)ix theory in the covering space \mathbf{T}_{5+n} of radii R_1, \dots, R_{5+n} and arrange a mirror parton to every zero-brane parton. In the small volume limit, the parton dynamics in the covering space is governed by $\mathcal{N} = 16$ supersymmetric $\text{U}(2N)$ Yang-Mills theory. The parameter space $\tilde{\mathbf{T}}_{5+n}$ on which the Yang-Mills theory is defined is dual to the covering space \mathbf{T}_{5+n} . Denote their radii as $\tilde{R}_1, \dots, \tilde{R}_{5+n}$. These radii and Yang-Mills gauge coupling constant are related to the M-theory parameters $R_1, \dots, R_{5+n}, R_{11}$ and ℓ_{11} as:

$$g_{\text{YM}}^2 = \ell_{11}^3 \left(\frac{\text{Vol}^{n+2}(\tilde{\mathbf{T}}_{5+n})}{\text{Vol}^3(\mathbf{T}_{5+n})} \right)^{\frac{1}{5+n}}, \quad \tilde{R}_M = \frac{\ell_{11}^3}{R_M R_{11}}. \quad (53)$$

Denote the $(5+n+1)$ dimensional fields as $A_M(x)$ ($M = 0, 1, \dots, 5+n$) for the gauge field, \mathbf{X}^i ($i = 5+n+1, \dots, 9$) for the pseudo-scalar adjoint scalar field and Θ_α for the 16-component spinor. We now construct M(atr)ix gauge theory relevant to $(\mathbf{T}_5/\mathbf{Z}_2) \otimes \mathbf{T}_n$ orbifold by identifying a suitable involution and projection thereof. Acting on the above fields, define a ‘parity’ transformation P defined by

$$\begin{aligned} P : \quad A_M(x) &\rightarrow P \cdot A_M(x) \cdot P^{-1} \equiv P_M^N A_N(P \cdot x) \\ \mathbf{X}^i(x) &\rightarrow P \cdot \mathbf{X}^i(x) \cdot P^{-1} \equiv -\mathbf{X}^i(P \cdot x) \\ \Theta_\alpha(x) &\rightarrow P \cdot \Theta_\alpha(x) \cdot P^{-1} \equiv \Gamma_\perp \Theta(P \cdot x). \end{aligned} \quad (54)$$

Here, the ‘parity’ matrix acting on $(5+n+1)$ -dimensional parameter space coordinates x^μ ’s is defined by

$$P_\mu^\nu = \text{diag.}(+, +, \dots, +, -, \dots, -), \quad (55)$$

where the negative signs are for 6-th through $(5+n)$ -th entries. The transformation coincides with the conventional definition of parity for $(5+n+1) = \text{odd}$ but not for even. We will nevertheless call the transformation as ‘parity’ transformation for lack of better notation. Note that we have taken \mathbf{X}^i ’s as pseudo-scalar fields, as is always the case for those originating from dimensional reduction. In addition, define an orientation reversal transformation Ω , whose local action on the fields is given by:

$$\begin{aligned} \Omega : \quad A_\mu(x) &\rightarrow \Omega \cdot A_\mu(x) \cdot \Omega^{-1} \equiv \pm A_\mu^T(x) \\ \mathbf{X}^i(x) &\rightarrow \Omega \cdot \mathbf{X}^i(x) \cdot \Omega^{-1} \equiv \pm \mathbf{X}^{iT}(x) \\ \Theta_\alpha(x) &\rightarrow \Omega \cdot \Theta_\alpha^T(x) \cdot \Omega^{-1} \equiv \pm \epsilon \Theta_\alpha^T(x). \end{aligned} \quad (56)$$

Here, the overall and the Bose-Fermi relative sign ambiguity is as before. The two-fold ambiguity (\pm) gives rise to $\text{USp}(2N)$ and $\text{SO}(2N)$ M(atr)ix gauge group for \pm sign choices respectively. Extending the argument in sections 2 and 3, we choose the M(atr)ix gauge group to be $\text{USp}(2N)$. The relative two-fold ambiguity ($\epsilon = \pm$) between the bosonic and the fermionic fields amounts to the interchange between the two eigenstates of Γ_\perp , hence, is fixed by the gamma matrix representations.

It is straightforward to see that combined action $\Pi \equiv \Omega \cdot P$ is an invariance of the covering space M(atr)ix gauge theory. Therefore, projecting out with $(1 + \Pi)/2$, we obtain M(atr)ix gauge theory for $(\mathbf{T}_5/\mathbf{Z}_2) \otimes \mathbf{T}_n$ orbifold compactification of the M theory. The resulting orbifold

M(atr)ix gauge theory has many exotic features not encountered in M(atr)ix gauge theory obtained after toroidal compactifications. First, from the action of P , we find that the parameter space on which the M(atr)ix gauge theory is defined is a dual orbifold $\tilde{\mathbf{T}}_5 \otimes (\tilde{\mathbf{T}}_n/\mathbf{Z}_2)$. The dual orbifold has 2^n fixed boundaries. Each boundary is given by five-dimensional dual torus $\tilde{\mathbf{T}}_5$. In fact, $(5+1)$ -dimensional chiral gauge theory on the boundary can be viewed as the intersection between $(5+p)$ -brane and $(9-p)$ brane.

5.2 Wilson Lines and Gauge Anomaly Cancellation

We have seen that the orbifold M(atr)ix gauge theory is defined on dual orbifold parameter space. Localized at the 2^n five-dimensional fixed boundaries are $(5+1)$ -dimensional $\mathrm{USp}(2N)$ chiral gauge theories. The thirty-two ‘half’ fundamental representation hypermultiplets identified as the twisted sector in the previous sections are then redistributed among the 2^n fixed boundaries. In this subsection, we show that transverse positions of the thirty-two ‘half’ fundamental representation hypermultiplets on the parameter space orbifold $\tilde{\mathbf{T}}_5 \otimes (\tilde{\mathbf{T}}_n/\mathbf{Z}_2)$ is the M(atr)ix gauge theory realization of turning on Wilson lines for the symmetry group associated with the sixteen longitudinal five-branes.

The situation is again exactly the same as the M(atr)ix theory on $(\mathbf{S}_1/\mathbf{Z}_2) \otimes \mathbf{T}_m$ orbifold [9]. Therefore, for comparison, we briefly recapitulate this case first. It has been shown that the M theory compactification dynamics is described by so-called heterotic M(atr)ix gauge theory on dual orbifold parameter space $\tilde{\mathbf{S}}_1 \otimes (\tilde{\mathbf{T}}_m/\mathbf{Z}_2)$. Localized at the 2^m fixed boundaries are $(1+1)$ -dimensional $\mathrm{SO}(2N)$ chiral gauge theories. The thirty-two Majorana-Weyl fermionic supermultiplets identified as the twisted sector are then redistributed among the 2^m fixed boundaries. Furthermore, it has been shown that locations of fermionic supermultiplets can be deformed away from the orbifold boundaries on the dual orbifold parameter space while maintaining gauge anomaly cancellation. A crucial ingredient for the deformation to be possible is the topological Wess-Zumino coupling of the M(atr)ix gauge theory to the background Ramond-Ramond fields and accompanying modification of background NS-NS fields. By detailed computation, it has been shown that the location deformation is nothing but the M(atr)ix gauge theory realization of turning on Wilson lines for $E_8 \times E_8$ gauge group of toroidally compactified heterotic string.

We thus begin with the role of topological Wess-Zumino coupling with background Ramond-Ramond fields for gauge anomaly cancellation via anomaly inflows. For N coincident D-branes, the Wess-Zumino coupling is given by

$$L_{\mathrm{WZ}} = \sum_{\mathrm{RR}} \int C_{\mathrm{RR}} \wedge \mathrm{Tr} \exp\left(\frac{iF}{2\pi}\right) \sqrt{\hat{A}(R)} \quad (57)$$

with

$$\sqrt{\hat{A}(R)} = 1 + \frac{1}{2}\hat{A}_4 + \left(\frac{1}{2}\hat{A}_8 - \frac{1}{8}\hat{A}_4^2\right) + \dots \quad (58)$$

Recall that the orbifold fixed boundaries carry nonzero Ramond-Ramond charges. The topological Wess-Zumino coupling Eq.(57) then tells us that the effective action of the orbifold M(atr)ix gauge theory is anomalous

$$D_\mu \frac{\delta \Gamma_{\mathrm{M}}[A]}{\delta A_\mu} = \frac{\mu_4}{2^m} \sum_{a=1}^{2^m} \int_{\tilde{\mathbf{T}}_5 \otimes \mathbf{R}} \mathrm{Tr}(F \wedge F \wedge F). \quad (59)$$

When the positions of the $\tilde{\mathbf{T}}_5$ hypersurfaces on which the ‘half’ fundamental representation hypermultiplets live are symmetrically located right at 2^n ($n = 1, \dots, 4$) dual orbifold boundaries, they induce gauge anomaly in such a way they cancel the gauge anomaly present at the orbifold boundaries. In other words, the Ramond-Ramond gauge fields are cancelled globally and there is no Wess-Zumino coupling present in the corresponding M(atrrix) gauge theory. Suppose we now move the twisted sector hypermultiplet locations are deformed away from the dual orbifold boundaries along $\tilde{\mathbf{T}}_m/\mathbf{Z}_2$ directions. In terms of underlying M-theory configurations, this deformation corresponds to turning on Wilson lines of the M(atrrix) gauge fields around the compactified \mathbf{T}_m toroidal directions.

The relevant Wess-Zumino couplings are

$$L_{\text{WZ}} = \int \left[C_2 \wedge \text{Tr}(F \wedge F \wedge F \wedge F) + C_6 \wedge \text{Tr}(F \wedge F) \right]. \quad (60)$$

Using the equation of motion for the Ramond-Ramond gauge fields, we find that these coupling gives rise to

$$D_\mu \frac{\delta \Gamma}{\delta A_\mu} = \sum \text{Tr}_{\mathbf{R}}(Q F \wedge F \wedge F) \quad (61)$$

under the gauge transformation. For the fundamental representation, we find that this is precisely the needed couplings.

With the gauge anomalies cancelled locally, it is then possible to deform the position of the twisted sector fundamental representation hypermultiplets. They are indeed in one to one correspondence with the Wilson lines that break the symmetry group associated with the sixteen longitudinal five-branes.

6 Discussion

In this paper, continuing the previous investigation, we have studied in detail M(atrrix) theory compactification on $\mathbf{T}_5/\mathbf{Z}_2$ orbifold. We have shown that the compactification is described by (0,8) supersymmetric $\text{USp}(2N)$ chiral gauge theory coupled to one untwisted sector hypermultiplet in anti-symmetric representation and thirty-two twisted sector hypermultiplets in ‘half’ fundamental representations. The gauge theory is defined on the dual parameter space $\tilde{\mathbf{T}}_5$. As the volume of the orbifold gets large, the dual parameter space becomes small. The M(atrrix) gauge theory then reduces to $\mathcal{N} = 8$ M(atrrix) quantum mechanics.

Similar to $\mathbf{S}_1/\mathbf{Z}_2$ orbifold case, the thirty-two fixed points of $\mathbf{T}_5/\mathbf{Z}_2$ turn out to carry anomalous G-flux. We have verified this from various consistency conditions. First parton scattering off orbifold fixed point has shown unambiguously the presence of -1/2 units of G-flux from each fixed point. Second, for shrinking orbifold, we have found that the (5+1)-dimensional $\text{USp}(2N)$ M(atrrix) gauge theory is free from both perturbative and global gauge anomaly only if we introduce thirty-two ‘half’ hypermultiplets. By pairing them we have found that the twisted sector hypermultiplets represent the sixteen longitudinal five-branes turned on $\mathbf{R}_{5,1}$.

One of the most interesting feature of M(atrrix) theory on $\mathbf{T}_5/\mathbf{Z}_2$ orbifold is that the required sixteen longitudinal five-branes can be arranged out of Landau-orbiting zero-branes. In other words, while introduced by hand the twisted sector at the start to meet the consistency

conditions, the required sixteen five-branes can be built out of zero-brane partons. This is in complete agreement with the idea of M(atrix) theory. Not only dynamical entities such as strings but also consistent compactification background can be built up out of zero-brane partons. This is the most distinguishing feature of $\mathbf{T}_5/\mathbf{Z}_2$ compared to other compactifications such as $\mathbf{S}_1/\mathbf{Z}_2$. The required longitudinal five-branes in the former are well-defined BPS states of the M(atrix) theory, while the required longitudinal nine-branes in the latter are not so, as has been shown in [20].

Furthermore, we have identified the $\mathbf{R}_{5,1}$ spacetime spectrum entirely within M(atrix) theory approach. We have found that the resulting massless excitations are in complete agreement with the six-dimensional (2,0) supergravity field content.

We have also shown, upon further toroidal compactification to $(\mathbf{T}_5/\mathbf{Z}_2) \otimes \mathbf{T}_m$ ($m = 1, 2, 3, 4$) orbifolds, M theory parton dynamics is described by orbifold M(atrix) gauge theory defined on a dual orbifold $\tilde{\mathbf{T}}_5 \otimes (\tilde{\mathbf{T}}_m/\mathbf{Z}_2)$. The toroidal compactification direction allows to turn on Wilson lines and trigger symmetry breaking for the twisted sector gauge group.

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